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Determination of single fibre strength distribution from fibre bundle testings

ZHIFAN CHI*, TSU-WEI CHOU, GUOYI SHEN†‡
Mechanical and Aerospace Engineering Department, and †Civil Engineering Department, University of Delaware, Newark, Delaware 19711, USA

The strength of fibres used as reinforcement materials for advanced composites is often assumed to follow the two-parameter Weibull distribution function. However, the experimental process widely used for obtaining the two parameters is tedious and prone to error. In this paper, two simple methods for determining the parameters of the Weibull distribution function are developed based upon the analysis of the tensile curves of fibre bundles. The first method focuses on the relation between the shape of a fibre bundle tensile curve and the survivability of fibres; the second method makes use of the relation between the maximum load point of a fibre bundle tensile curve and the shape parameter of the Weibull distribution of fibre strength. These two methods, in particular the second one, have greatly simplified the fibre testing process. Experimental results on Thornel-300 carbon fibres further demonstrate the validity of these techniques.

1. Introduction

Fibres used as reinforcement materials for advanced composites, such as carbon and glass, exhibit high tensile strength. On the other hand, the presence of defects, especially on fibre surfaces, is responsible for the variations in fibre strength. Thus, it is pertinent in the analysis and prediction of composite strength to take into account not only fibre average strength but also fibre strength distributions. The large amount of test data accumulated on fibre strength measurements indicates that it is feasible to represent fibre strength distributions by a two-parameter Weibull distribution function [1–3].

A procedure often adopted by researchers in determining fibre strength distribution is through the measurement of the average strength of a group of fibres of the same length. The Weibull shape parameter is then determined from the relationship between fibre average strength and gauge length [4]. There are shortcomings in such measurements. First, it is rather tedious to extract individual fibres from a bundle and to perform numerous tests on fibres with very small diameter. Second, the extraction of fibres from a bundle inevitably has "selected" the stronger ones, since the weaker fibres are prone to damage and fracture in the process. Third, it is almost impossible to determine the exact cross-sectional area of a single fibre. Experiments based upon laser diffraction fringes have shown that the measured fibre diameters vary along the fibre length due to fibre twist and the non-circular fibre cross-section [5].

Manders and Chou [5] discussed some of the problems experienced in their measurements of fibre strength. They examined single fibre strength by measuring fibre bundle strength. The work was restricted to the determination of Weibull shape parameters only.

In this paper, the strength of single fibres is assumed to follow the two-parameter Weibull distribution. A theoretical expression of the load–strain (P–ϵ) relationship for a bundle of fibres under tension has been derived first. Then, two methods for determining the two parameters of Weibull distribution for single fibre strength have been developed. This is done by analysing the characteristics of the P–ϵ expression. The validity

*On leave from Northwestern Polytechnical University, Xian, Shaanxi, China.
‡On leave from Yantze Valley Planning Office, Wuhan, China.

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of these methods are verified by experimental measurements of carbon fibre strength.

2. Analysis

2.1. Assumptions

The following assumptions are basic to the analytical work: (1) the distribution of single fibre strength under tension follows the two-parameter Weibull distribution,

$$F(\sigma) = 1 - \exp[-L(\sigma/\sigma_0)^m] \quad (1)$$

where $F(\sigma)$ is the failure probability of single fibres of length L under an applied stress no greater than σ, σ_0 and m denote the scale parameter and shape parameter of the Weibull distribution, respectively; (2) the relationship between applied stress, σ, and strain, ϵ, for a single fibre follows Hooke's Law up to fracture,

$$\sigma = E_f\epsilon \quad (2)$$

where E_f is the fibre elastic modulus; (3) the applied load is distributed uniformly among the surviving fibres at any instant during a bundle tensile test.

2.2. Fibre bundle tensile load–strain (P–ϵ) relationship

From Equations 1 and 2, we obtain

$$F(\epsilon) = 1 - \exp[-L(\epsilon/\epsilon_0)^m] \quad (3)$$

where $F(\epsilon)$ is the failure probability of a single fibre under strain no greater than ϵ, ϵ_0 is the scale parameter of Weibull distribution for strain, and

$$\epsilon_0 = \sigma_0/E_f \quad (4)$$

At an applied strain, ϵ, the number of surviving fibres in a bundle, which consists of N_0 fibres, is

$$N = N_0[1 - F(\epsilon)] = N_0\exp[-L(\epsilon/\epsilon_0)^m] \quad (5)$$

N can be related to the applied tensile load, P, on the bundle by

$$P = \sigma AN = AE_f\epsilon N_0\exp[-L(\epsilon/\epsilon_0)^m] \quad (6)$$

Equation 6 is the relationship of P–ϵ for a bundle of fibres under tension, where A is the cross-sectional area of a single fibre. If the A, N_0, L, E_f, ϵ_0 and m are known, the P–ϵ curve for a bundle of fibres could be drawn according to Equation 6. Fig. 1 shows such an example.

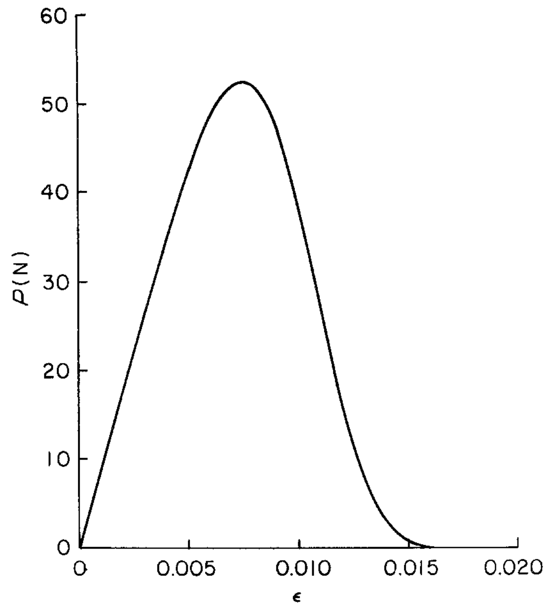

Figure 1 Theoretical P–ϵ curve for carbon fibre, $E_f = 225$ GPa, $d_f = 7\ \mu$m, $N_0 = 1000$, $m = 4.5$ and $\epsilon_0 = 0.026$.

2.3. Characteristics of the theoretical P–ϵ curve

2.3.1. The shape of the P–ϵ curve

For brittle solids, the P–ϵ curve usually terminates at the point of P_{max}. However, it can be seen from Equations 6 and 7 in the following and Fig. 1 that in the case of a brittle bundle the P–ϵ curve is continuous and smooth everywhere. After reaching the point P_{max}, the load decreases gradually to zero.

2.3.2. The slope of the P–ϵ curve at $\epsilon = 0$

From Equation 6,

$$\frac{dP}{d\epsilon} = AE_fN_0[1 - Lm(\epsilon/\epsilon_0)^m]\exp[-L(\epsilon/\epsilon_0)^m] \quad (7)$$

and the slope at $\epsilon = 0$ is

$$S_0 = \left.\frac{dP}{d\epsilon}\right|_{\epsilon=0} = AE_fN_0 \quad (8)$$

Thus the equation for the tangent line of the P–ϵ curve at $\epsilon = 0$ is

$$P^* = AE_fN_0\epsilon \quad (9)$$

2.3.3. The relationship between a P–ϵ curve and the survivability of single fibres

It is observed from Equations 6 and 9 that

$$\frac{P}{P^*} = 1 - F(\epsilon) \quad (10)$$

or

$$\frac{P}{S_0\epsilon} = 1 - F(\epsilon) \quad (11)$$

The survivability of fibres at a given level of ϵ on the P–ϵ curve can be determined by Equation 11. Thus, Equation 11 forms the base of the first method presented in the next section for determining the parameters of fibre strength distribution.

2.3.4. Maximum load on the P–ϵ curve
The strain corresponding to the maximum load on the P–ϵ curve is obtained from Equation 7 at $dP/d\epsilon = 0$ as

$$\epsilon_{\max} = \epsilon_0 \left(\frac{1}{Lm}\right)^{1/m} \quad (12)$$

Thus, the maximum load is

$$P_{\max} = AN_0E_f\epsilon_0\left(\frac{1}{Lme}\right)^{1/m} \quad (13)$$

where $e = 2.71828\ldots$.

From Equations 8, 12 and 13, the slope of the straight line connecting the origin and the point of $P_{\max}$ is given by

$$S_A = \frac{P_{\max}}{\epsilon_m} = S_0\left(\frac{1}{e}\right)^{1/m} \quad (14)$$

or

$$m = 1/\ln\left(\frac{\epsilon_m S_0}{P_{\max}}\right) \quad (15)$$

Equation 15 is the base of the second method presented in the next section for determining the parameters of fibre strength distribution

3. Single fibre strength distributions
Two methods for determining the distribution of single-fibre strength are presented in this section.

3.1. The first method
The procedure, as based upon Equation 11, is outlined below.

1. Determine the values of P and ϵ of a P–ϵ curve obtained from the tensile test of a loose bundle.

2. Calculate the slope S_0 by Equation 8 with the data E_f, A and N_0 of the fibre bundle.

3. Determine from Equation 11 the fibre survivability, $1 - F(\epsilon)$, corresponding to each strain of the experimental P–ϵ curve for the fibre bundle.

4. The experimental data points of $1 - F(\epsilon)$ are plotted against ϵ on a Weibull probability paper. If all the data points are situated close to a straight line, the strength of fibres can be assumed to follow the Weibull distribution, and the shape parameter, m, for the fibre strength distribution is obtained from the slope of the straight line.

5. The scale parameter, ϵ_0, is determined from Equation 12.

3.2. The second method
Based on Equation 15, the following procedure is followed:

1. obtain the $P_{\max}$ and ϵ_m from the experimental P–ϵ curve for fibre bundles;

2. calculate initial slope, S_0, of the curve from Equation 8;

3. obtain the shape parameter, m, from Equation 15; and

4. the scale parameter, ϵ_0, is determined from Equation 12.

It is interesting to note that a theoretical P–ϵ curve can be constructed according to the values of A, E_f, N_0, L, m and ϵ_0 obtained by this method (see Equation 6). By comparing the theoretical and experimental P–ϵ curves, the reliability of the assumptions involved in the analysis can be assessed.

4. Experimental analysis
In order to examine the reliability of the two methods developed in this paper, the strength distribution of Thornel-300 carbon fibres is measured using two conventional methods as well as the two new methods outlined in Section 3.

First, the strengths of single-fibre specimens (600 mm gauge length) are measured. The shape parameter, m, is determined by measuring the slope of the straight line fitting the test data points plotted on a Weibull probability paper. The value of $m = 4.7$ is obtained. Then the average strengths, $\bar{\sigma}$, of a single fibre of gauge lengths 10, 30 and 60 mm are measured, respectively. The shape parameter is obtained by measuring the slope ($-1/m$) of the straight line fitting the data points on a plot of $\log\bar{\sigma}$ against $\log L$. The value of $m = 6.6$ is obtained in this case. The detail of the experiments and the analysis of the data are described by Chi and Chou [6].

The load–strain (P–ϵ) curves for bundles of Thornel-300 carbon fibres ($N_0 = 1000$, fibre diameter $= 7\,\mu$m, $E_f = 225$ GPa) with gauge length of 60 mm are measured. The shape parameter, m, and the scale parameter, ϵ_0, are obtained by using the two new methods of Section 3 for each speci-

TABLE I Values of m and ϵ_0 obtained by method 1

Specimen number	m	ϵ_0
1	4.56	0.0277
2	4.06	0.0301
3	4.38	0.0257
4	4.45	0.0249
5	6.54	0.0166
6	4.08	0.0227
7	4.08	0.0324
Average	4.6	0.026

men. The data are given in Tables I and II. The average shape parameter is 4.6 and 4.5 for the methods of Sections 3.1 and 3.2, respectively. These are very close to the value of m of 4.5 obtained from the experiment for single fibres with gauge length of 60 mm. The scale parameter is 0.026 for both new methods.

The assumption that the fibre strength follows the Weibull distribution is further substantiated in Figs. 2 and 3. Fig. 2 shows that all the experimental points of specimen 3 follow nearly a straight line when plotted on Weibull probability paper. This implies that the strength distribution of Thornel-300 carbon fibre obeys the two-parameter Weibull distribution function.

Theoretical P–ϵ curve with parameters obtained from specimen 3 with the method given in Section 3.2 is shown in Fig. 3 by the solid curve. The experimental data points are also indicated. The consistency between the theory and experiment is excellent in the range of bundle strain not much greater than ϵ_m.

5. Discussions

The theoretical P–ϵ curves for fibre bundles, according to Equation 6, should be smooth curves as shown in Fig. 1. However, the existence of the step-wise decrease in load (Fig. 4) at a constant strain can be explained as follows. When a single

TABLE II Values of m and ϵ_0 obtained by method 2

Specimen number	m	ϵ_0
1	3.86	0.0332
2	3.48	0.0362
3	4.51	0.0249
4	4.55	0.0244
5	5.39	0.0195
6	4.33	0.0214
7	5.54	0.0236
Average	4.5	0.026

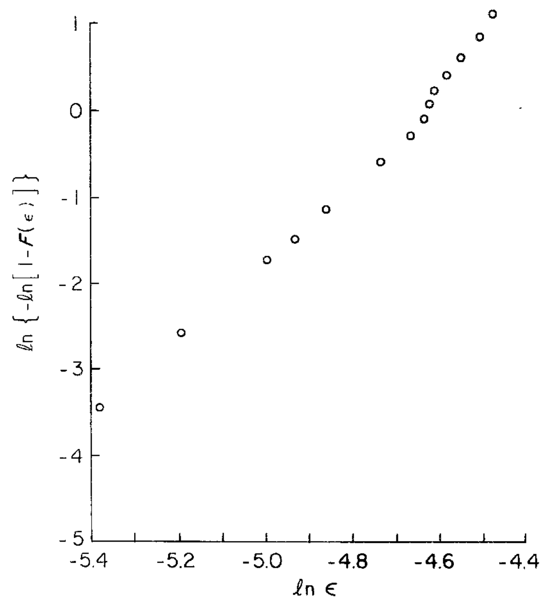

Figure 2 Strength distribution of single fibres obtained from the tensile curve of a fibre bundle on Weibull probability paper.

fibre in a bundle breaks, the load originally carried by this fibre is transferred to the surviving fibres. This sudden increase in load will promote more fibre breakages in the bundle than in the case where the unbroken fibres are under static loading. Thus, a certain number of fibres will break in a very short period of time and so the small

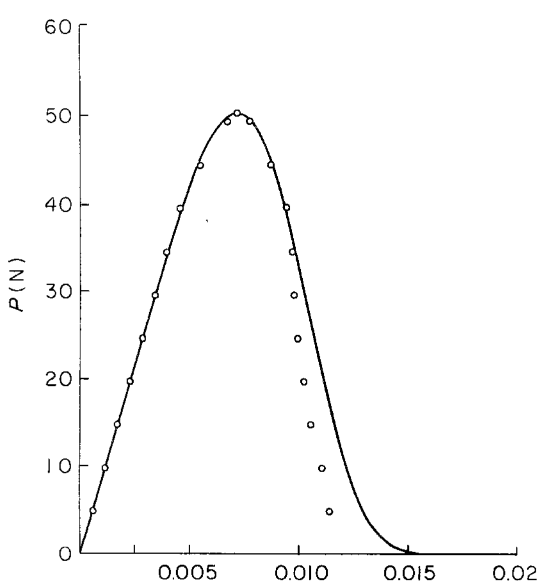

Figure 3 Comparison of a theoretical P–ϵ curve with experimental data.

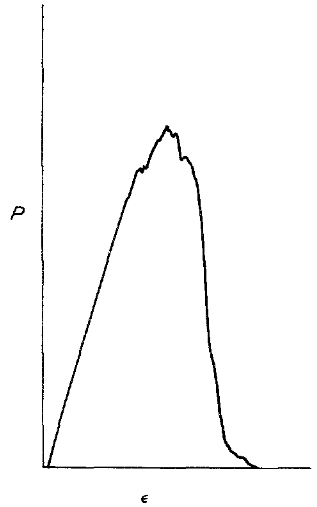

Figure 4 A typical P–ϵ curve from experiment.

decreases of load in the P–ϵ curves appear. If the bundle contains more fibres, the decreases in load should be less and smaller such as described by Manders and Chou [5], where a bundle contains 1200 fibres. In order to facilitate the analysis, the P–ϵ curve is replaced by its outer envelope.

The difference between the theoretical and experimental P–ϵ curves increases in the unloading part of the P–ϵ curve (Fig. 3). The strain at a given applied load in the experimental P–ϵ curve is smaller than that in the theoretical curve. This discrepancy may be partially attributed to the dynamic load during fibre breakage. The influence of the sudden breakage of a fibre is different when it happens in different stages of the P–ϵ curve. The influence is smaller in the initial stage, but the influence is larger in the unloading stage since there are less surviving fibres to share the dynamic load. We do not consider the influence of dynamic load when developing the theoretical expression.

The tensile strain, ϵ, can be determined by measuring the relative displacement of two end tabs on a specimen with a displacement gauge. However, in the present experiments, the measured displacement is not the relative displacement of the two end tabs but an absolute displacement of the crosshead of the testing machine. The major contribution to this relative displacement occurs between the end tabs and the grips and it must be deducted from the measured absolute displacement. It has been assumed in the present experiments that the relative displacements between the end tabs and the grips are linearly proportional to the applied load during the loading process, and the relative displacements remain constant after the load reaches the P_{max} point during the unloading process.

Sufficiently large numbers of bundle tests should be performed in order to accurately determine the fibre Weibull parameters. For the example given in this paper, the results obtained from seven fibre bundle tests are fairly close to those obtained from single fibre tests, where the number of specimens is 56 [6].

It could be seen from the example that the value of m of 4.5 to 4.6 obtained from the bundle is close to the m value of 4.7 obtained from the single fibre tensile testing with the same gauge length as that of the tested bundle. However, these value are different from the m value of 6.6 obtained from the $\log \bar{\sigma}_f$–$\log L$ plot for single fibre tested with different gauge length. The same phenomenon has been reported by Manders and Chou [5], whose m value of 11 obtained from the $\log \bar{\sigma}_f$–$\log L$ plot is much larger than the m value of 4.3 to 5.3 obtained from other methods.

In Equation 1, the influence of fibre length on the fibre strength distribution is expressed by the exponent L. The shape parameter, m, is independent of the fibre length. ϵ_0 is the scale parameter corresponding to a fibre of unit length and the unit adopted here is a millimetre. The values of parameters determined in this paper can be applied along with Equation 1 to fibres of different lengths.

6. Conclusions

1. The theoretical expression of the plot of load against tensile strain for a loose bundle of fibres, of which the strength follows the Weibull distribution function, has been established as Equation 6. The theoretical curve, as shown in Fig. 1, is a continuous and smooth curve under both loading and unloading conditions. The experimental curves exhibit steps, i.e. decreases in tensile load at constant strain, which are attributed to the dynamic load effect.

2. Two methods for determining the Weibull parameters of single-fibre strength distribution have been developed based on the analysis of the properties of the fibre bundle tensile curve.

The first method utilizes Equations 11 and 12. The fibre survivability, $1 - F(\epsilon)$, corresponding to each value of ϵ is determined by Equation 11 using the P and ϵ values of the experimental fibre bundle tensile curve. The shape parameter, m, can then be determined from the slope of a straight line, which fits the experimental points ($1 - F(\epsilon)$ against ϵ) plotted on a Weibull probability paper. Then the scale parameter, ϵ_0, can be obtained by Equation 12.

The second method is based on Equations 15 and 12. The shape parameter, m, can be determined from Equation 15 with the values of P_{max} and ϵ_m obtained from the experimental fibre bundle tensile curve. Then the scale parameter, ϵ_0, can also be determined from Equation 12.

3. Experimental determinations of the Weibull parameters for single-fibre strength have shown that the present methods are simpler to perform as compared to the direct measurement of single-fibre strength. Also, reliable data can be acquired with relatively small numbers of tests.

4. Comparisons of the theoretical P–ϵ curve with experimental data have shown that for a given strain in the unloading process the fibre bundle loading measured experimentally is much smaller than that of the theoretical value. This observation implies that the frictional force between the broken and unbroken fibres, if it exists, is insignificant.

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